Advance
KENTUCKY

AP Calculus Prep Session Handout

Riemann Sums and Trapezoidal Approximations

(Numerical Approximations to the Definite Integral)

Riemann Sums and Trapezoidal Approximations

Numerical approximations of the definite integral, which includes Riemann sums and trapezoidal approximations, appear regularly on the AP examination in both free response and multiple-choice. When these problems appear on the examination, they are usually part of a problem with a table given as part of the stem. Students should not depend on a formula which assumes that the subintervals within the sum are equal. Similarly, students should not depend on using a graphing calculator program since these questions are often on the no calculator portion of the exam, often with unequal subintervals which makes most of such programs difficult, if not impossible, to use.

Students should also be familiar with defining a definite integral as a limit of a Riemann sum. This is sometime tested on the multiple-choice as a limit of a sum that students are expected to recognize as a definite integral.

What students should know how to do:

- Approximate definite integrals using right-hand, left-hand, and midpoint Riemann sums. (These are the *only* types of Riemann sums tested.) Remember, subintervals may be equal or unequal.
- Approximate definite integrals using a trapezoidal approximation. Since subintervals may be unequal, students should not be limited to memorizing the Trapezoid Rule.
- Determine whether a numerical approximation to a definite integral is greater than or less than the actual value of the integral.
- Identify the limit of a Riemann sum as a definite integral (usually tested on multiple-choice).

In the free-response questions that follow, the parts of the question that directly related to numerical approximations of the definite integral have been highlighted.

Free Response.

2007 AB5 BC5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.3	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \leq t \leq 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)dt$? Give a reason for your answer.

2004 AB 3 BC 3 (Form B)

t (minutes)	0	5	10	15	20	25	30	35	40
$v(t)$ (miles per minute)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time *t* minutes, where *v* is a differentiable function of *t*. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

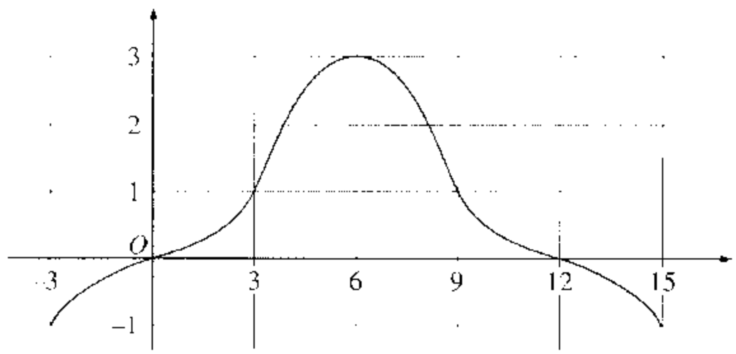
(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function *f*, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{10}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicate units of measure.

(d) According to the model *f*, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

2002 AB 4 and BC 4 Form B



Graph of f

The graph of a differentiable function *f* on the closed interval $[-3,15]$ is shown in the figure above. The graph of *f* has a horizontal tangent line at $x=6$. Let $g(x)=5+\int_6^x f(t)\,dt$ for $-3 \le t \le 15$.

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is *g* decreasing? Justify your answer.

(c) On what intervals is the graph of *g* concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

2001 AB 2 and BC 2

t (days)	$W(t)$ $(°C)$
0	20
3	31
6	28
9	24
12	22
15	21

The temperature, in degrees Celsius $(°C)$, of the water in a pond is a differentiable function *W* of time *t*. The table above shows the water temperature as recorded every 3 days over a 15-day period.

(a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \le t \le 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.

(c) A student proposes the function *P*, given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time *t*, where *t* is measured in days and *P*(*t*) is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.

(d) Use the function *P* defined in part (c) to find the average value, in degrees Celsius, of *P*(*t*) over the time interval $0 \le t \le 15$ days.

1999 AB 3 and BC 3

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function *R* of time *t*. The table above shows the rate as measured every 3 hours for a 24-hour period.

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$.
Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period. Indicate units of measure.

Multiple-Choice

Part A. No Calculator Allowed.

x	2	5	10	14
$f(x)$	12	28	34	30

1. The function f is continuous on the closed interval [2, 14] and has values as shown in the table above. Using the subintervals [2, 5], [5, 10], and [10, 14], what is the approximation of $\int_2^{14} f(x)dx$ found by using a right Riemann sum?

   (A) 296 (B) 312 (C) 343 (D) 374 (E) 390

2. The expression $\frac{1}{50}\left(\sqrt{\frac{1}{50}}+\sqrt{\frac{2}{50}}+\sqrt{\frac{3}{50}}+\cdots+\sqrt{\frac{50}{50}}\right)$ is a Riemann sum approximation for

   (A) $\int_0^1 \sqrt{\frac{x}{50}}\,dx$

   (B) $\int_0^1 \sqrt{x}\,dx$

   (C) $\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}\,dx$

   (D) $\frac{1}{50}\int_0^1 \sqrt{x}\,dx$

   (E) $\frac{1}{50}\int_0^{50} \sqrt{x}\,dx$

3. The closed interval $[a, b]$ is partitioned into n equal subintervals, each of width Δx, by the numbers $x_0, x_1, \dots, x_n$ where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. What is $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\Delta x$?

   (A) $\frac{2}{3}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

   (B) $\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

   (C) $\frac{3}{2}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

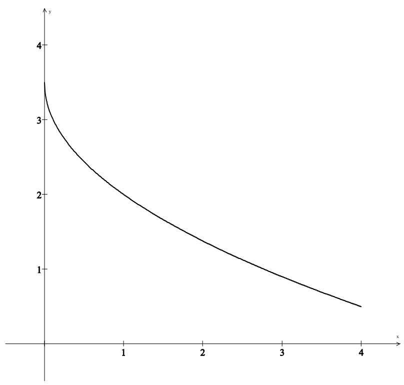
   (D) $\left(b^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$

   (E) $2\left(b^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$

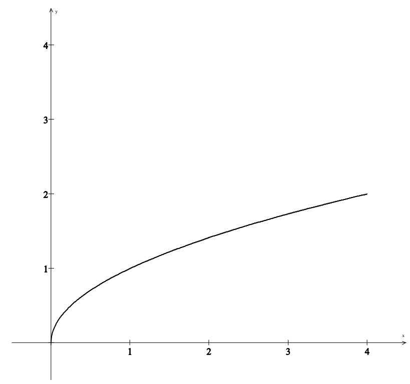
Part B. Graphing Calculator Allowed.

4. If a trapezoidal sum overapproximates $\int_0^4 f(x)dx$, and a right Riemann sum underapproximates $\int_0^4 f(x)dx$, which of the following could be the graph of $y = f(x)$?

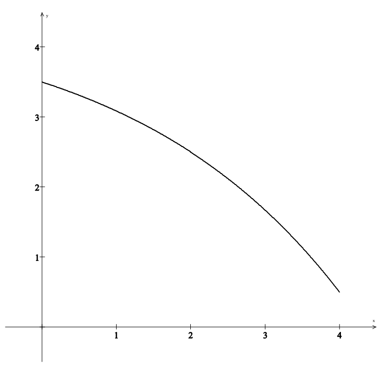
(A)



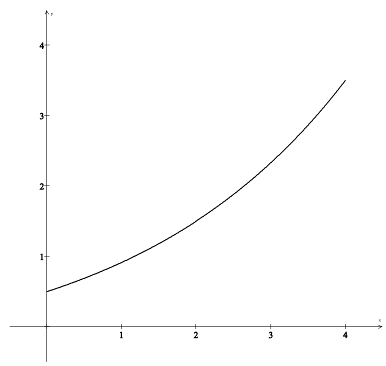
(B)



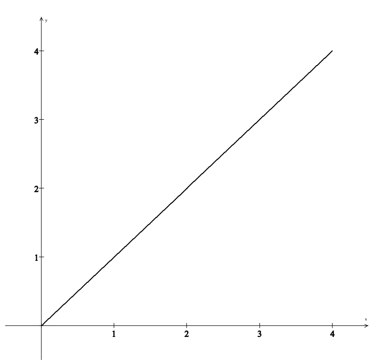
(C)



(D)



(E)



x	2	5	7	8
$f(x)$	10	30	40	20

5. The function f is continuous on the closed interval [2, 8] and has values that are given in the table above. Using the subintervals [2, 5], [5, 7], and [7, 8], what is the trapezoidal approximation of $\int_2^8 f(x)dx$?

(A) 110 (B) 130 (C) 160 (D) 190 (E) 210

t (sec)	0	2	4	6
$a(t)$ (ft/sec^2)	5	2	8	3

6. The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at $t = 0$ is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

(A) 26 ft/sec (B) 30 ft/sec (C) 37 ft/sec (D) 39 ft/sec (E) 41 ft/sec

x	0	0.5	1.0	1.5	2.0
$f(x)$	3	3	5	8	13

7. A table of values for a continuous function f is shown above. If four equal subintervals of [0, 2] are used, which of the following is the trapezoidal approximation of $\int_0^2 f(x)dx$?

(A) 8 (B) 12 (C) 16 (D) 24 (E) 32